Lecture 3

1. Hilbert Spaces
2. The general notion of the basis
3. The Weierstrass Theorem
4. The generalization of the Weierstrass Thm

Lecture 3

1. Hilbert Space: $L^2_w(a,b)$.

Let $f \in L^2_w(a,b)$, and let $|e_i\rangle$, $i=1,2,\dots$ be a basis of $L^2_w(a,b)$ then.

$$|f\rangle = \sum_{i=1}^{\infty} f^i |e_i\rangle$$

where f^i, $i=1,2,\dots$ called Fourier coefficients.

There is a 1-1 correspondence between the function belonging to $L^2_w(a,b)$ and ~~the correspon~~ their Fourier coefficients.

$$|f\rangle \longleftrightarrow (f^1, f^2, \dots, f^n, \dots)$$

such that

$$\sum_{i=1}^{\infty} |f^i|^2 < \infty$$

The space of the Fourier coefficients is known as ℓ^2. This is also a Hilbert space. We can find the induced inner product of ℓ^2. Defining

$$|\bar{f}\rangle = (f^1, f^2, \dots, f^n, \dots)$$

we can show that

$$\langle \bar{f} | \bar{g} \rangle = \sum_{i=1}^{\infty} \bar{f}_i g_i \quad , \quad L^2_w(a,b) \text{ and } \ell^2 \text{ are isomorphic.}$$

2. The general notion of the basis of $L^2_\omega(a,b)$ (The generalization of the notion of basis).

Given any sequence

$$|g_1\rangle, |g_2\rangle, \dots, |g_k\rangle, \dots \quad (1)$$

of linearly independent vectors, we can obtain a sequence of orthogonal vectors

$$|e_1\rangle, |e_2\rangle, \dots, |e_k\rangle, \dots \quad (2)$$

where $|e_k\rangle$ is a linear combination of all $|g_i\rangle$ for each $i \le k$. The procedure to do this is the Gram-Schmidt orthogonalization (See appendix)

In the general case the transformed set $|e_k\rangle$ will not be a basis of $L^2_\omega(a,b)$. We have the following theorem.

<u>Lemma</u>: The sequence of orthonormal vectors (2) obtained by the orthogonalization of linearly independent vectors (1) is a basis of the space if and only if each vector $|f\rangle \in L^2_\omega(a,b)$ is a limit vector of a sequence of a sequence of linear combinations of vectors $|g_i\rangle$, $i = 1, 2, \dots$

Proof:

i) Suppose an arbitrary vector $|f\rangle \in L^2_w(a,b)$ which is a limit vector of some sequence $|a_k\rangle$ of linear combinations of vectors $|g_i\rangle$, $|a_k\rangle = \sum_{i=1}^{k} b^{(k)}_i |g_i\rangle$. On the other hand by the Gram-Schmidt method we can obtain an orthonormal basis $|e_k\rangle$ as a linear combination of $|g_i\rangle$, $i \leq k$ and hence $|g_k\rangle$'s can be written as a linear combination of $|e_i\rangle$ $(i \leq k)$. Hence $|f\rangle$ is a ~~linear~~ limit vector of the sequence

$$|a_k\rangle = \sum^{k} b^{(k)}_i |g_i\rangle = \sum^{k} a^i_{(k)} |e_i\rangle$$

where $k = 1, 2, \ldots$

By assumption

$$\rho(|f\rangle, |a_k\rangle) \xrightarrow[k \to \infty]{} 0$$

It can be verified that

$$\rho^2(|f\rangle, |a_k\rangle) = (\langle f| - \langle a_k|)(|f\rangle - |a_k\rangle)$$

$$= \langle f|f\rangle - \langle f|a_k\rangle - \langle a_k|f\rangle + \langle a_k|a_k\rangle$$

$$= \langle f|f\rangle - \sum a^i_{(k)} \langle f|e_i\rangle - \sum \bar{a}^i_{(k)} \langle e_i|f\rangle + \sum |a^i_{(k)}|^2$$

$$= \langle f|f\rangle - \sum_{i=1}^{k} |\langle f|e_i\rangle|^2 + \sum_{i=1}^{k} |a^i_{(k)} - \langle e_i|f\rangle|^2$$

RHS ≥ 0 First term is positive due to the Bessel inequality.

as $k \to \infty \Rightarrow \langle f|f\rangle = \sum |f^i|^2$ hence

$\langle e_i\rangle$'s are basis. (Parseval's thm)

ii) If the vectors $|e_i\rangle$ $(i = 1, 2, \ldots)$ form a basis of the space, then according to to theorem that there exists a sequence $|f_k\rangle = \sum f^i |e_i\rangle$ converges to $|f\rangle$.

Remark: Hence it may be regarded as the limit vector of a sequence of linear combination of the vectors $|g_k\rangle$

Basis: An orthonormal set $|e_i\rangle$ is a basis if any vector $|f\rangle \in L^2_w(a,b)$ can be expressed as a linear combination of $|e_i\rangle$'s $i = 1, 2, \ldots$

3. The Weierstrass Theorem.

Let $|1\rangle, |2\rangle, \ldots, |n\rangle, \ldots \in L^2_w(a,b)$.

representing the functions

$$1, x, x^2, \ldots, x^{n-1}, \ldots$$

form a basis of $L^2_w(a,b)$

<u>Theorem:</u> Let the function $f(x)$ be continuous on the finite, closed interval $[a,b]$. For any $\varepsilon > 0$, there exist a positive integer n and a corresponding polynomial $P_n(x)$ of the nth degree such that

$$|f(x) - P_n(x)| \leq \varepsilon, \quad \text{for any } x \in [a,b]$$

- ε is independent of x. An arbitrary continuous function f defined on a closed, finite interval approximate uniformly by suitable polynomials

$$f \simeq P_n(x)$$

- Any continuous function can be approximated <u>uniformly</u> and with any desired accuracy by linear combination of the functions.

. For the proof of the thm please see OK

4. Generalization of the Weierstras Theorem.

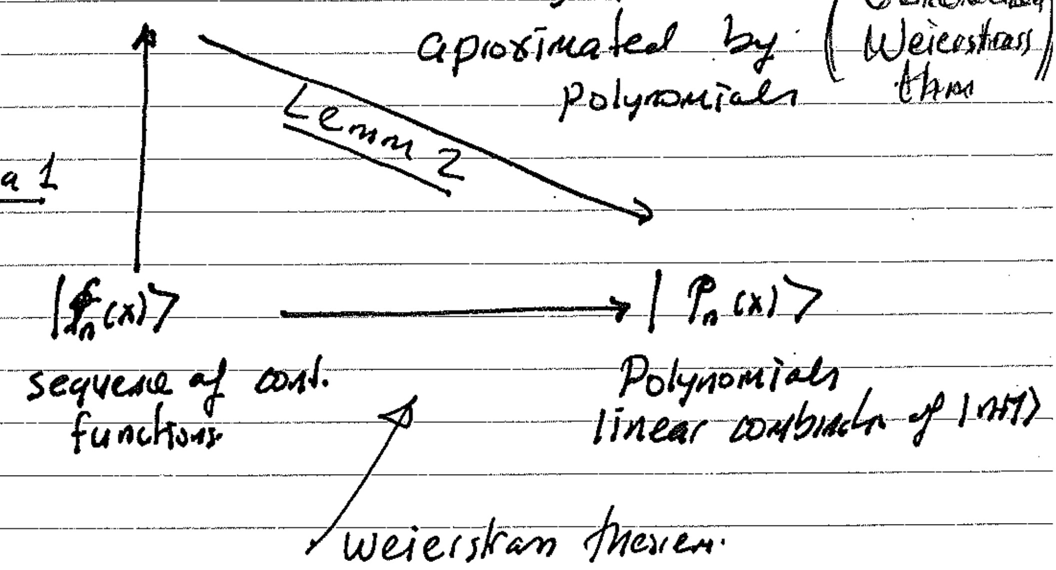
If $f(x) \in L^2_w(a,b)$ and if the polynomials $p_n(x)$, $n=0,1,\dots$ represented by the vectors $|p_n\rangle$, $n=0,1,\dots$ which are linear combination of the ~~first~~ $(n+1)$-vectors converges uniforomly to $f(x)$.

This means that $\rho(|f\rangle, |p_n\rangle)$ can be made arbitrarily small by the proper choice of $|p_n\rangle$. f is approximated by polynomials.

<u>Lemma 1</u>: Let $f \in L^2_w(a,b)$, there exists a sequence of vectors $|f_k\rangle$ that can be represented by continuous functions ~~and~~ converges to $|f\rangle$
(we shall not prove it)

<u>Lemma 2</u>: Any function $f \in L^2_w(a,b)$ can be approximated by polynomials. $p_n(x)$
linear combinations of $|n\rangle = 1, x, \dots, x^{n-1}, x^n$.

<u>proof</u>:



$$\rho(|f\rangle, |P_k(x)\rangle) \le \rho(|f\rangle, |f_n\rangle) + \rho(|f_n\rangle, |P_n\rangle)$$
$$\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$$\rho(|f\rangle, |f_n\rangle) \le \varepsilon/2$$
$$\rho(|f_n\rangle, |P_k\rangle) \le \varepsilon/2.$$

Hence we have

Theorem: Let $f \in L_w^2(a,b)$, then it is represented by a polynomial, Hence it is a linear combination of $|k\rangle$, $k=1,2,..$

Hence $|k\rangle$, $k=1,2,..$ is a basis of $L_w^2(a,b)$.

SET 2

MATH 543: ORTHOGONAL FAMILIES AND BASIS

1. Prove that the Fourier coefficients of any $|f>\in L^2_w(a,b)$ form a Hilbert space. First prove that the space of such coefficients form an inner product space then prove that this inner product space is complete. This space is $\mathbf{l_2}$ (see set 1). Prove that l_2 and $L^2_w(a,b)$ are isomorphic. See page 196 of DK.
2. Prove that all finite dimensional inner product spaces are complete. See page 182-183 of DK.
3. Explain the importance of Bessel's inequality and Parseval's relation.
4. Assume that there exists an orthonormal basis $|e_i>$, $(i=1,2,\cdots)$ in $L^2_w(a,b)$. Then , for any $|f>\in L^2_{a,b}$, the sequence of vectors

$$|f_k>=\sum_{i=1}^{k} f^i\,|e_i>$$

with

$$f^i=<e_i|f>$$

has $|f>$ as the limit vector in the sense that

$$\lim_{k\to\infty}\rho(|f>,|f_k>)=0$$

5. Prove that the set of orthonormal vectors $|e_i>$ form a basis of $L^2_e(a,b)$ if and only if the Fourier coefficients wrt $|e_i>$ satisy the Parseval's relation.
6. Prove that any orthogonal family $|e_i>$, $\;i=1,2,=\cdots$ in $L^2_w(a,b)$ is linearly independent
7. If $(e_1,e_2,\cdots,e_n)$ is a finite orthonormal family of finctions in $L^2_w(a,b)$ and $f\in L^2_w(a,b)$, then

$$||f-\sum_{k=1}^{n} f^k\,|e_k>||$$

has its minimum value when $f^k = < f|e_k >$ and

$$\sum_{k=1}^{n} | < f|e_k > |^2 \le ||f||^2$$

8. Let $f, g \in L^2_w(a,b)$ prove that $| < f|g > | < \infty$.

9. (Gram Schmidt Orthogonalization Process (GS)). Let $|g_i >$, $i = 1, 2, \cdots$ be linearly independent set of vectors in $L^2_w(a,b)$. We can construct an orthogonal set $|u_i >$, $(i = 1, 2\cdots)$ from the following (prove this statement)

$$|u_1 >= |g_1 >, \quad |u_i >= |g_i > - \sum_{k=1}^{i-1} \alpha_k \, |u_k > \tag{1}$$

where $\alpha_k = \frac{<g_k, u_k>}{||u_k||^2}$ for $k > 1$. Then the orthonormal set is given by $|e_k >= \frac{|u_k>}{||u_k||}$

10. Let $I = [-1, 1]$, $w = 1$ and $(g_i) = (1, x, x^2, \cdots)$ be the linearly independent set in $L^2(-1, 1)$. Find the orthonormal set $|e_i >$ obtained from this linearly independent set. This set is Called the Legendre polynomials.

11. Prove the following proposition: The sequence of orthonormal vectors $\{|e_i >\}$ obtained by the orthogonalization (GS Method) of a linearly independent vectors $|g_i >$ of the space $L^2_w(a,b)$ is a basis of the space if and only if each vector $|f >\in L^2_w(a,b)$ is a limit vector of a sequence of linear combinations of the vectors $|g_i >$.

12. In $L^2_w(a,b)$ any linearly independent set of vectors ($g_i >$'s in the previous problem) may be considered as a basis of this space if an arbitrary vector of $L^2_w(a,b)$ can be expressed as a limiting vector of linear combination of basis vectors. Prove that the set $\{1, x, x^2, \cdots\}$ form a basis of continuous functions in $[a,b]$.

13. Let $f >\in L^2_w(a,b)$ can be expressed as a limiting vector of a sequence of vectors that can be expressed as a linear combination of continuous functions. Prove that , using this aasertion and the previous problem, any $|f >\in L^2_w(a,b)$ can be approximated by suitable polynomials. This means

that given $\varepsilon > 0$ there exists a positive integer m and a polynomial p_m such that $||f - p_m|| < \varepsilon$

14. Definition: A complete orthonormal sequence $\{|e_i>, \ i = 1, 2, \cdots\}$ in the space $L^2_w(a,b)$ is called a basis of $L^2_w(a,b)$. Prove the following proposition. Let $\{|e_i>\}, \ i = 1, 2, \cdots\}$ be an orthonormal sequence in $L^2_w(a,b)$. The following statements are equivalent:

1. $\{|e_i>, i = 1, 2, \cdots\}$ is complete.
2. $|f> = \sum_{k=1}^{\infty} <e_k|f> \ |e_i>$, for all $|f> \in L^2_w(a,b)$
3. $\sum_{k=1}^{\infty} |e_i> < e_k| = \mathbf{1}$.
4. $<f|g> = \sum_{k=1}^{\infty} <f|e_k> < e_k|g>$. for all $|f>, |g> \in L^2_w(a,b)$.
5. $||f||^2 = \sum_{k=1}^{\infty} | <e_k|f> |^2$, for all $|f> \in L^2_w(a,b)$.

Hint:. Use the following directions $1 \to 2 \to 3 \to 4 \to 5 \to 1$.

Lecture4

1) Classical Orthogonal Polynomials
2) Classification of the Classical Polynomials
3) Recursion Relations
4) Differential Equation satisfied by Classical Polynomials.
5) G-S method.
6) Expansion of Functions in series of orthogonal Functions
7) Some examples of orthogonal polynomials

Lecture 4.

Classical Orthogonal Polynomials:

- According to the generalized Weierstrass theorem, we learned that the set of functions

$$1, x, x^2, \ldots, x^n, \ldots$$

forma a basis of $L^2_w(a,b)$

- Using the GS method with different w, and $[a,b]$ one can construct all such Polynomials
- We shall not follow this procedure

① Generalized Rodriguez formula

Let us consider the set of ~~polynomials~~ function in x

$$C_{(n)}(x) = \frac{1}{w}\frac{d^n}{dx^n}(w s^n) \qquad (n=0,1,2,\ldots)$$

where $C_{(1)}(x)$, $w = w(x)$, and $s(x)$ satisfy the following conditions:

(i) $C_{(1)}(x)$ is a first degree polynomial in x
(ii) $s(x)$ is a polynomial in x of degree ≤ 2 with real roots.
iii) $w(x)$ is real, positive, and integrable in $[a,b]$ and satisfies the boundary conditions

$$w(a)\, s(a) = w(b)\, s(b) = 0$$

Theorem: $C_{(n)}(x)$ is a polynomial of the nth degree, which is orthogonal on the interval $[a,b]$, with weight $w(x)$, to any polynomial $P'_{(m)}(x)$ of degree $m < n$

$$\int_a^b P_{(m)}(x)\, C_{(n)}(x)\, w(x)\, dx = 0 \qquad (m<n).$$

Before proving this theorem we prove the following Lemmas:

Lemma 1: Denote $P_{(\leq k)}(x)$ as an arbitrary polynomial of degree less or equal to k. Then the following holds

$$\frac{d^m}{dx^m}\left(w s^n P_{(\leq k)}\right) = w s^{(n-m)} P_{(\leq k+m)}$$

Proof: ① $C_{(1)}(x) = \frac{1}{w}\frac{d}{dx}(ws) = \frac{ds}{dx} + \frac{s}{w}\frac{dw}{dx}$

RHS is a polynomial of the first degree.

② $m=1$

$$\frac{d}{dx}\left(w s^n P_{(\leq k)}\right) = \frac{dw}{dx} s^n P_{(\leq k)}$$

$$+ n w \frac{ds}{dx} s^{n-1} P_{(\leq k)}$$

$$+ w s^n \frac{dP_{\leq k}}{dx}$$

$$= w s^{n-1}\left\{\left[\frac{s}{w}\frac{dw}{dx} + n\frac{ds}{dx}\right] P_{(\leq k)} + s\frac{dP_{\leq k}}{dx}\right\}$$

$$= w s^{n-1} P_{(\leq k+1)}$$

(3) induction: Assume that

$$\frac{d^m}{dx^m}\left(w s^n P_{(\le k)}\right) = w s^{n-m} P_{(\le k+m)}$$

holds prove that it holds also for $m+1$.
Differentiation:

$$\frac{d^{m+1}}{dx^{m+1}}\left(w s^n P_{\le(k)}\right) = \frac{d}{dx}\left[w s^{n-m} P_{(\le k+m)}\right]$$

use (2) here

$$= w s^{n-m-1} P_{(\le k+m+1)}$$

$$= w s^{n-(m+1)} P_{(\le k+(m+1))}$$

QED.

<u>Lemma 2</u>: All derivatives $\frac{d^m}{dx^m} w s^n$ with $m<n$ vanish at $x=a$ and $x=b$.

<u>proof</u>: Take $k=0$ and $P_{(\le k)}(x) = 1$.

$\Rightarrow$ From the previous lemma we get

$$\frac{d^m}{dx^m}\left(w s^n\right) = w s^{n-m} P_{\le m}(x)$$

since $n>m$. RHS becomes zero at $x=a$ and $x=b$ for all m $(m<n)$.

We are ready now to prove Theorem.

by Lemma 1 $C_{(n)}(x)$ is a polynomial of degree $\leq n$

i) Let us first show that

$$\langle P_{(m)}(x), C_{(n)}(x) \rangle = 0 \qquad m < n$$

$$= \int_a^b P_{(m)}(x) \frac{d^n}{dx^n}[w s^n]\, dx$$

$$= \cancel{P_{(m)}(x) \frac{d^{n-1}}{dx^{n-1}}(w s^n) \Big|_a^b}^{\,0} - \int_a^b \frac{d}{dx} P_{(m)}(x) \frac{d^{n-1}}{dx^{n-1}}$$

$$= - \int_a^b \frac{d}{dx} P_m(x) \frac{d^{n-1}}{dx^{n-1}} w s^n \, dx$$

proceeding this (n-times) we obtain

$$= (-1)^{n+1} \int_a^b \left(\frac{d^n}{dx^n} P_m(x) \right) w s^n \, dx$$

but $\frac{d^n}{dx^n} P_m(x) = 0$ since $n > m$

$$\Rightarrow \langle P_{(m)}(x), C_{(n)}(x) \rangle = 0 \qquad m < n$$

ii) $C_{(n)}$'s are Polynomials of degree n

We can write (Since $C_{(n)}$ is of degree $\leq n$)

$$C_{(n)}(x) = a_{(n)} x^n + P_{(\leq n-1)}(x)$$

Here $a_{(n)}$ is some real number which $\neq 0$

The norm of $C_{(n)}$

$$\| C_n(x) \|^2 = \int_a^b C_n(x)^2 \, w(x) \, dx$$

$$= \int_a^b C_n(x) \left(\frac{d^n}{dx^n} w s^n \right) dx$$

$$= \int_a^b w(x) \left[a_n x^n + \cancel{P_{(\leq n-1)}(x)}^{\,0} \right] C_n(x) \, dx$$

$$= a_n \int w(x) \, x^n \, C_n(x) \, dx$$

LHS > 0, hence $a_n \neq 0$. This proves that C_n's are of degree n.

Hence we have

$$\int_a^b w(x) \, C_{(n)}(x) \, C_{(m)}(x) \, dx = 0 \qquad m \neq n.$$

$$m = n \qquad \| C_n \|^2 = \int_a^b w(x) \, C_{(n)}^2(x) \, dx \neq 0$$

Orthogonal set of Polynomials. To make this set orthonormal we can introduce a new constant in the Rodriquez formula

$$C_{(n)}(x) = \frac{1}{K_n w} \frac{d^n}{dx^n} w s^n \qquad n = 0, 1, 2, \ldots$$

$C_{(0)}, C_{(1)}, \ldots$ form an orthogonal set on $[a, b]$

② Classification of the Classical Polynomials

$C_{(1)}(x)$ is a first-degree polynomial.
We can always, by a suitable choice of the arguement x we can define

$$C_{(1)}(x) = -\frac{x}{K_1}$$

Remember that

$$C_{(n)}(x) = \frac{1}{K_n w} \frac{d^n}{dx^n} w s^n$$

$$\Rightarrow \quad C_{(1)}(x) = \frac{1}{K_{(1)} w} \frac{d}{dx} w s = -\frac{x}{K_1}$$

$$\Rightarrow \quad \frac{d}{dx} w s = -x w.$$

$$\frac{dw}{w\,dx} = -\frac{x + ds/dx}{s}.$$

$$\boxed{\frac{w'}{w} = -\frac{1}{s}(x+s').}$$

s is a polynomial of degree ≤ 2

The idea is to determine w, s and $[a,b]$ from the above equation.

i) Assume that $s = \alpha$ constant. $\Rightarrow w'/w = -\frac{x}{\alpha}.$

$w = \text{cont.}\, e^{-x^2/2\alpha}$ to satisfy the BCs.

$a = -\infty$, $b = \infty$

Rescaling: $s=1$, $w=e^{-x^2}$ $[-\infty,\infty]$

$C_n(x)=\frac{1}{K_n}e^{x^2}\frac{d^n}{dx^n}e^{-x^2}$ "Hermite Polynomials"

ii) Take $s(x)=\beta(x-\alpha)$ $\Rightarrow$

$$w(x)=\text{const.}\,(x-\alpha)^{\left(-\frac{\alpha+\beta}{\beta}\right)}e^{-x/\beta}$$

if $\beta>0$, $\nu\equiv-\frac{\alpha+\beta}{\beta}>-1$ $(-\alpha/\beta>0)$

$\Rightarrow$ $s(x)\,w(x)$ vanish at $x=\alpha$ and $x=\infty$

and $w(x)$ integrable in the integrable $[\alpha,\infty)$.

making a linear transform

$(x-\alpha)/\beta\to x$ and rescaling

we get

$$s=x,\quad w(x)=x^{\nu}e^{-x}\quad \nu>-1\quad [0,\infty)$$

iii) $s(x)=\gamma(x-\alpha)(\beta-x)$ $\beta>\alpha$. (Laguerre Polyn.)

again linear transform and rescaling we get:

$$s=1-x^2,\quad w(x)=(1-x)^{\nu}(1+x)^{\mu}\quad \mu,\nu>-1$$

$[a=-1,\ b=1]$ "Jacobi Polynomials"

Remarks: 1. if $s(x)$ has a double root
The boundary condition are not satisfied.

2. If $s(x)$ has imaginary roots
again the BC not satisfied.

Table

Interval	weight func.	s(x)	Name of the Polynomial
$(-\infty, \infty)$	e^{-x^2}	1	Hermite, $H_n(x)$
$[0, \infty)$	$x^{\nu} e^{-x}$ $(\nu > -1)$	x	Laguerre, $L_n^{\nu}(x)$
$[-1, 1]$	$(1-x)^{\nu}(1+x)^{\mu}$ $(\mu, \nu > -1)$	$1-x^2$	Jacobi, $P_n^{(\nu,\mu)}(x)$

Depending on the real numbers μ, ν Jacobi Polynomial have some special names

Interval	weight func.	s(x)	Name of the polyn.
$[-1, 1]$	$(1-x^2)^{\lambda - 1/2}$	$1-x^2$	Gegenbauer, $C_n^{\lambda}(x)$
$[-1, 1]$	1	$1-x^2$	Legendre, $P_n(x)$
$[-1, 1]$	$(1-x^2)^{-1/2}$	$1-x^2$	Tchebichef of the first kind $T_n(x)$
$[-1, 1]$	$(1-x^2)^{1/2}$	$1-x^2$	Tchebichef of the second kind $U_n(x)$

$$C_{(n)}(x) = \frac{1}{K_n w} \frac{d^n}{dx^n} (w s^n).$$

Standardization. K_n "Orthonormalization"

③ The recursion relation:

$$C_{(n+1)}(x) = (A_n x + B_n)\, C_{(n)}(x) + D_n C_{n-1}(x)$$

A_n, B_n, and D_n are some constants (not depending on x) depending on n.

Notation: $k_n \equiv$ coefficient of x^n in $C_{(n)}(x)$

$k_n' \equiv$ coefficient x^{n-1} in $C_{(n)}(x)$

$$C_{(n)}(x) = k_n x^n + k_n' x^{n-1} + P_{(n-2)}(x)$$

$$h_n \equiv \int_a^b C_{(n)}^2\, w\, dx = k_n \int_a^b x^n C_{(n)}\, w\, dx$$

and we shall make use of the orthogonality relations

$$\int_a^b C_{(n)}(x)\, P_{(m)}(x)\, w(x)\, dx = 0 \qquad m < n$$

Consider the following expression:

$$C_{(n+1)}(x) - \frac{k_{n+1}}{k_n}\, x\, C_{(n)}(x)$$

which clearly of degree $\leq n$. and therefore can be written as

$$C_{(n+1)}(x) - \frac{k_{n+1}}{k_n}\, x\, C_{(n)}(x) = \sum_{\ell=0}^{n} a_{(\ell)}^{(n)}\, C_{(\ell)}(x)$$

It is easy to see that all

$a^{(n)}_{(\ell)} = 0 \qquad \ell = 0, 1, 2, \ldots, n-2$

multiply both sides by $C_{(m)}(x)$ and integrate over $[a, b]$ for $m < n-2$

$$LHS = 0 \qquad RHS = \sum' a^{n}_{(\ell)} \int w(x)\, C_{(m)}(x)\, C_{(\ell)}(x)\, dx$$

$$= 0$$

$$a^{n}_{(m)} \cdot h_m = 0 \qquad \forall\, m \leq n-2.$$

$$\text{since } h_m \neq 0 \Rightarrow a^{(n)}_{(m)} = 0 \qquad m \leq n-2$$

$\Rightarrow$

$$C_{(n+1)}(x) - \frac{k_{n+1}}{k_n} x\, C_{(n)}(x) = a^{(n)}_{(n)}\, C_{(n)} + a^{(n)}_{(n-1)}\, C_{(n-1)}$$

i) multiply both by $w\, C_{(n-1)}$ and integrate

$$a^{(n)}_{(n-1)}\, h_{n-1} = -\frac{k_{n+1}}{k_n} \int_a^b x\, C_{n-1}\, C_{(n)}\, w\, dx$$

$$= -\frac{k_{n+1}}{k_n}\, k_{n-1} \int_a^b x^n\, C_{(n)}\, w\, dx$$

$$= -\frac{k_{n+1} \cdot k_{n-1}}{k_n} \cdot \frac{h_n}{k_n}$$

$$\boxed{a^{(n)}_{(n-1)} = -\frac{k_{n-1}\, k_{n+1}}{k_n^2}\, \frac{h_n}{h_{n-1}}}$$

ii) compare the coeff. of x^{n-1} both side

$$k_{n+1}' - \frac{k_{n+1}}{k_n} k_n' = a^{(n)}_{(n)} k_n$$

$$a^{(n)}_{(n)} = \frac{k_{n+1}'}{k_n} - \frac{k_{n+1}}{k_n^2} k_n'$$

$$\Rightarrow \quad A_n = \frac{k_{n+1}}{k_n}, \quad B_n = a^{(n)}_{(n)}, \quad D_n = -a^{(n)}_{(n-1)}$$

(4) D.E satisfied by Classical Polynomials

$$\frac{1}{w}\frac{d}{dx}\left(w s \frac{d C_{(n)}}{dx}\right)$$ by lemma 1 is a polynomial of degree $\leq n$.

$$= -\sum_{l=1}^{n} \lambda^{(l)}_{(n)} C_{(l)}$$

multiply by $w\, C_{(m)}(x)$ $\quad m < n$.

$LHS = 0 \Rightarrow RHS \quad \lambda^{(l)}_{n} = 0 \quad l < n$.

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$$\Rightarrow \frac{d}{dx}\left(w s \frac{d C_{(n)}}{dx}\right) = -\lambda^{(n)}_{(n)}\, w\, C_{(n)}(x)$$

$$= -\lambda_n\, w\, C_{(n)}(x)$$

$$\lambda_n = -n\left[K_1 \frac{d C_{(1)}}{dx} + \frac{1}{2}(n-1)\frac{d^2 s}{dx^2}\right]$$

① $$\int_a^b C_{(n)} \frac{d}{dx}\left[s w \frac{d C_{(n)}}{dx} \right] dx = -\lambda_n h_n$$

$$= \int_0^b C_n \left[\frac{d}{dx}(sw)\, C_n' + sw\, C_n'' \right] dx$$

$$= \int_0^b w\, C_n \left[K_1 C_1 \frac{dC_n}{dx} + s \frac{d^2 C_{(n)}}{dx^2} \right] dx$$

$$C_n = k_n x^n + P_{(\leq n-1)}(x)$$

$$= \int_0^b w\, C_n \left[n k_n K_1 C_1 x^{n-1} + s\, n(n-1) k_n x^{n-2} \right]$$

$$= \left[n K_1 \frac{dC_1}{dx} + \frac{1}{2} n(n-1) \frac{d^2 s}{dx^2} \right] \underbrace{\int_0^b w\, C_n\, (k_n x^n)\, dx}_{h_n}$$

$$\lambda_n = -n \left[K_1 \frac{dC_1}{dx} + \frac{1}{2} n(n-1) \frac{d^2 s}{dx^2} \right]$$

~~[illegible]~~ Another common property of classical Polynomials (8)

(1) Prove that classical polynomials $C_n(x)$ with $x \in I \subset \mathbb{R}$ has n-distinct zeros in I.

proof: Let $P_m(x)$ $(m < n)$ be a polynomial of x then we have

$$\int_I w(x)\, C_n(x) P_m(x)\, dx = 0 \qquad m < n$$

hence we get $\int_I w(x)\, C_n(x)\, dx = 0$, $(\forall n)$ (P_0 = nonzero constant). This means that $C_n(x)$ changes its sign in I. Let us say that there are k number of zeros in I $(k < n)$. Let

$$C_n(x) = q(x)\, h_k(x), \quad h_k(x) \equiv \prod_{i=1}^{k} (x - x_i)$$

where $q(x)$ is a polynomial of degree $n-k$ which doesnot change its sign in I. It is clear that

$$\int_I w(x)\, C_n(x)\, h_k(x)\, dx = 0 \qquad \forall k < n$$

but the integrand is positive definite (or negative definite) in I. This is a contradiction. Hence $k = n$. There are n number of zeros in I. To show they are distinct, let us assume that one of the roots, x_k has multiplicity $\ell \geq 2$

$$C_n = (x - x_k)^{\ell} \prod_{i \neq k}^{n-\ell} (x - x_i) = (x - x_k)^{\ell} S_k(x)$$

consider the integral

$$\int_I w(x)\, (x - x_k)^{\varepsilon} S_k\, C_n(x)\, dx =$$

where $\varepsilon = 0$ if ℓ = even and $\varepsilon = 1$ if ℓ = odd. This integral is zero for all $\ell \geq 2$, but the integrand is positive definite in I. Once again a contradiction. Hence the only possibility $\ell = 1$ which mean that all zeros must be distinct

(6) Expansion of Functions in series of Orthogonal Polynomials

$$|e_i\rangle = \frac{1}{\sqrt{h_i}}\,|C_{(i)}\rangle \quad \in L_w^2(a,b)$$

form an orthonormal basis of $L_w^2(a,b)$

Any function $|f\rangle$ in $L_w^2(a,b)$ has the form

$$|f\rangle = \sum_{i=0}^{\infty} f^i \frac{1}{\sqrt{h_i}}\,|C_{(i)}\rangle$$

where $\quad f^i = \frac{1}{\sqrt{h^i}}\langle C_{(i)}|f\rangle \qquad i=0,1,2,\ldots$

Define a sequence of functions (sequence of partial sum)

$$f_{(n)} = \sum_{i=0}^{n} f^i \frac{1}{\sqrt{h^i}}\,|C_{(i)}\rangle$$

converge in the mean to $f(x)$

$$\lim_{n\to\infty}\int_a^b \left| f(x) - f_{(n)}\right|^2 w(x)\,dx = 0$$

for finite intervals : It is true also for Hermite and Laguerre Polynomials.

Legendre Polynomials $P_n(x)$

Standardization: $K_n = (-2)^n n!$

Constants:

$$k_n = \frac{2^n \Gamma(n+1/2)}{n!\,\Gamma(1/2)}, \quad k_n' = 0$$
$$h_n = (n+1/2)$$

Rodriguez formula:

$$P_n(x) = \frac{(-1)^n}{2^n n!} \frac{d^n}{dx^n} (1-x^2)^n$$

Differential Eqn.

$$(1-x^2)\frac{d^2}{dx^2} P_n(x) - 2x \frac{d}{dx} P_n(x) + n(n+1) P_n(x) = 0$$

Recurrence formula

$$(n+1) P_{n+1}(x) = (2n+1)\, x\, P_n(x) - n P_{n-1}(x)$$

Some of them are

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2 = \frac{1}{2}(3x^2-1)$$
$$P_3(x) = \frac{1}{2}(5x^3-3x), \quad P_4(x) = \frac{1}{8}(35x^4-30x^2+3), \ldots$$

$$P_n(x) = (-1)^n P_n(-x), \quad P_n(1) = 1$$

Orthogonal Polynomials

Hermite

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$

$$k_n = 2^n, \quad h_n = \sqrt{\pi}\, 2^n n!, \quad K_n = (-1)^n$$

$$H_{n+1} = 2x H_n - 2n H_{n-1}, \quad \underline{H_n' = 2n H_{n-1}}$$

$$H_n'' - 2x H_n' + 2n H_n = 0$$

$$\frac{1}{w} \frac{d}{dx}\left(w s \frac{d}{dx} C_n\right) = \lambda_n C_n$$

$$w = e^{-x^2}, \quad s = 1, \quad \lambda_n = -2n$$

Legendre Polynomials $P_n(x)$

$$K_n = (-1)^n 2^n n!, \quad w = 1$$

$$P_n(x) = \frac{(-1)^n}{2^n n!} \frac{d^n}{dx^n}\left[(1-x^2)\right]$$

$$k_n = \frac{2^n \Gamma(n+1/2)}{n!\,\Gamma(1/2)}, \quad k_n' = 0, \quad \lambda_n = -n(n+1)$$

$$(1-x^2)\frac{d^2}{dx^2} P_n(x) - 2x \frac{dP_n}{dx} + n(n+1) P_n = 0$$

$$(1-x^2) P_n' + n x P_n - n P_{n-1} = 0$$

$$P_{n+1}' - x P_n' - (n+1) P_n = 0$$

$$P_{n+1}' - P_{n-1}' - (2n+1) P_n = 0$$

~~5th Lecture (6 Oct 2013)~~

~~1) Recursion relations~~
~~2) Differential equations~~
~~3) Some orthogonal polynomials~~
~~4) Expansion of Functions in series of Orthogonal Polynomials~~

Some Examples of orthogonal polynomials

(3)

$[a,b]$	w	s	Polynomial
$(-\infty,\infty)$	e^{-x^2}	1	Hermite, $H_n(x)$
$[0,\infty)$	$x^{\nu}e^{-x}$ $(\nu>-1)$	x	Laguerre, $L_n^{\nu}(x)$
$[-1,1]$	$(1-x)^{\nu}(1+x)^{\mu}$ $(\mu,\nu>-1)$	$1-x^2$	Jacobi, $P_n^{(\mu,\nu)}(x)$

(i) Hermite Polynomials $H_n(x)$.

Standardization: $K_n=(-1)^n$, $w=e^{-x^2}$

Constants: $k_n=2^n$, $k_n'=0$, $h_n=2^n n!\sqrt{\pi}$

Rodrigues formula $H_n(x)=(-1)^n e^{x^2}\dfrac{d^n}{dx^n}e^{-x^2}$

Differential Eqn.

$$\frac{d^2}{dx^2}H_n(x)-2x\frac{d}{dx}H_n(x)+2n H_n(x)=0$$

Recurrence formula

$$H_{n+1}(x)=2x H_n(x)-2n H_{n-1}(x).$$

$H_0=1$, $H_1=2x$, $H_2=e^{x^2}\frac{d}{dx}(2xe^{-x^2})=2-4x^2$, ...

$$\int_{-\infty}^{\infty}H_1H_2e^{-x^2}dx=4\int_{-\infty}^{\infty}e^{-x^2}x(1-2x^2)\,dx$$

$$\int_0^{\infty}+\int_{-\infty}^{0}=\int_0^{\infty}+\int_{\infty}^{0}=0 \quad \text{identically}$$

~~A~~

$\Rightarrow$ Gram-Schmid Procen for orthogonalizing a linearly indepent sequence $|g_i\rangle$, $i=1,2,\ldots$ in an inner product space. The resulting orthonormal sequence $|e_i\rangle$ has property that.

$$\text{Span}\{|e_1\rangle, |e_2\rangle, \ldots, |e_n\rangle\} = \text{Span}\{|g_1\rangle, |g_2\rangle, \cdots |g_n\rangle\}$$

1) The first element $|e_1\rangle$ is

$$|e_1\rangle = \frac{1}{\|g_1\|}|g_1\rangle$$

2) let $|v_2\rangle = |g_2\rangle + \alpha|e_1\rangle$

since $\langle e_1|v_2\rangle = 0$

$$\alpha = -\langle e_1|g_2\rangle$$

$$\Rightarrow |v_2\rangle = |g_2\rangle - (\langle e_1|g_2\rangle)|e_1\rangle$$

$$\Rightarrow |e_2\rangle = \frac{1}{\|v_2\|}|v_2\rangle = \frac{1}{\|v_2\|}(|g_2\rangle - \langle e_1|g_2\rangle|e_1\rangle)$$

3) Similarly.

$$|v_3\rangle = |g_3\rangle + \beta|e_2\rangle + \gamma|e_1\rangle$$

orthonality relations $\langle e_1|v_3\rangle = \langle e_2|v_3\rangle = 0$ give

$$\gamma = -\langle e_1|g_3\rangle, \quad \beta = -\langle e_2|g_3\rangle$$

$$\Rightarrow |e_3\rangle = \frac{|v_3\rangle}{\|v_3\|}$$

$$\Rightarrow \quad |e_3\rangle = \frac{1}{\|v_3\|} \left[|g_3\rangle - \langle e_2|g_3\rangle |e_2\rangle - \langle e_1|g_3\rangle |e_1\rangle \right]$$

4) Hence in general we have

$$|e_n\rangle = \frac{1}{\|v_n\|} |v_n\rangle$$

where

$$|v_n\rangle = |g_n\rangle - \sum_{i=1}^{n-1} \langle e_i|g_n\rangle |e_i\rangle$$

For infinite dimensional case we let $n \to \infty$

Application of GS when $w=1$, $[a,b]=[-1,1]$
and $\{|g_i\rangle\} = \{1, x, x^2, \dots, x^n, \dots\}$

i) $|g_0\rangle = 1$, $\|g_0\|^2 = \int_{-1}^{1} 1 \cdot dx = 2$

$|e_0\rangle = \frac{1}{\sqrt{2}}$, $v_0 = 1$

ii) $v_1 = x - \langle e_0|x\rangle \cdot 1$

$\langle e_0|x\rangle = \frac{1}{\sqrt{2}} \int_{-1}^{1} x \, dx = 0$

$v_1 = P_1(x) = x$

$e_1 = \frac{P_1(x)}{\|v_1\|}$, $\|v_1\| = \int_{-1}^{1} x^2 dx = \frac{2}{3}$

$= \sqrt{\frac{3}{2}}\, x$

iii) $v_2 = x^2 - \langle e_1|x^2\rangle x - \langle e_0|x^2\rangle \frac{1}{\sqrt{2}}$

$\langle e_1|x^2\rangle = \sqrt{\frac{3}{2}} \int_{-1}^{1} x\, x^2 dx = 0$

$\langle e_0|x^2\rangle = \frac{1}{\sqrt{2}} \int_{-1}^{1} x^2 dx = \frac{2}{3\sqrt{2}}$

$v_2 = P_2 = x^2 - \frac{1}{3}$

- - -

$V_0 = 1 \longrightarrow P_0(x) = 1$

$V_1 = x \longrightarrow P_1(x) = 1$

$V_2(x) = x^2 - \frac{1}{3} \rightarrow P_2(x) = \alpha\left(x^2 - \frac{1}{3}\right)$

$P_2(1) = 1 \Rightarrow \alpha = \frac{3}{2} \Rightarrow P_2(x) = \frac{1}{2}(3x^2 - 1)$

$\Rightarrow V_3 = e_3 - \cancel{\langle e_2 | g_3 \rangle e_2}^{=0} - \langle e_1 | g_3 \rangle e_1 - \cancel{\langle e_0 | g_3 \rangle e_0}^{=0}$

$= x^3 - \langle e_1 | g_3 \rangle e_1 \quad , \quad e_1 = \sqrt{\frac{3}{2}}\, x$

$\langle e_1 | g_3 \rangle = \int_{-1}^{1} \sqrt{\frac{3}{2}}\, x \cdot x^3 dx = \frac{2}{5}\sqrt{\frac{3}{2}}$

$\Rightarrow V_3(x) = x^3 - \frac{3}{5}x \Rightarrow P_3(x) = \alpha\left(x^3 - \frac{3}{5}x\right)$

$P_3(1) = 1 \Rightarrow \alpha = 5/2$

$P_3(x) = \frac{1}{2}(5x^3 - 3x)$

— — —

we obtain the Legendre polynomial this way

$\int_{-1}^{1} P_n(x) P_m(x)\, dx = 0 \quad m \neq n$

$P_n(1) = 1. \quad \forall n$

⇒ Hermite polynomials in Physics

- Harmonic oscillator in Quantum Mechanics

The Schrodinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}m\omega^2x^2\psi = E\psi$$

one dimensional problem. A particle with mass m under the influence of potential $V(x) = \frac{1}{2}m\omega^2x^2$

ψ is the wave function which should be normalizable

let $\xi = \sqrt{\frac{m\omega}{\hbar}}\, x \Rightarrow$

$$\psi'' + (K-\xi^2)\psi = 0, \quad K = \frac{2E}{\hbar\omega}$$

Letting $\psi = e^{-\xi^2/2} h(\xi) \Rightarrow$

$$\frac{d^2h}{d\xi^2} - 2\xi\frac{dh}{d\xi} + (K-1)h = 0$$

if $K \neq 1 + 2n$, n integer

$h \sim e^{\xi^2} \rightarrow \psi$ is not normalizable

The only solution which is physical is

that

$$K-1 = 2n \Rightarrow E = (n+1/2)\hbar\omega$$

Energy is quantized

n positive integers

$$\frac{d^2h}{d\xi^2} - 2\xi\frac{dh}{d\xi} + 2nh = 0 \Rightarrow h = H_n(\xi).$$

$$\Rightarrow \quad \psi(x) = e^{-x^2/2} H_n(\xi).$$

The norm of ψ in $\mathcal{H}$.

$$\|\psi\|^2 = \int_{-\infty}^{\infty} \psi^*(x)\,\psi(x)\,dx$$

$$= \int_{-\infty}^{\infty} e^{-x^2} H_n(x)^2\,dx$$

$$= \|\psi\|^2_{L^2}.$$

Appendix

SET 3

MATH 543: ORTHOGONAL POLYNOMIALS

1. **Generating Functions:** For the classical orthogonal polynomials $C_n(x)$ we have seen so far there exists a generating function $g(x,t)$ for each defined by

$$g(x,t) = \sum_{n=0}^{\infty} a_n\, t^n\, C_n(x), \tag{1}$$

where a_n,s are some real numbers. Find these numbers for the following cases

Plynomial $C_n(x)$	Generating Function $g(x,t)$
Hermite, $H_n(x)$	e^{-t^2+2xt}
Laguerre, $L_n^{\nu}(x)$	$e^{-xt/(1-t)}/(1-t)^{\nu+1}$
Legendre, $P_n(x)$	$(t^2-2xt+1)^{-1/2}$
Chebyshev (I) , $T_n(x)$	$(1-t^2)(t^2-2xt+1)^{-1}$
Chebyshev (II), $U_n(x)$	$(t^2-2xt+1)^{-1}$

2. Find or prove the following for the Legendre polynomials by using its generating function where $a_n = 1$:
(a) $P_n(-x) = (-1)^n P_n(x)$, (b). $||P_n||$, (c). $\int_0^{\infty} P_n(x)dx$,
(d)$(1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)\,P_n(x) = 0$,
(e) $(1-x^2)\,P_n'(x) + nxP_n(x) - nP_{n-1}(x) = 0$,
(f) $P_{n+1}'(x) - P_{n-1}'(x) - (2n+1)P_n(x) = 0$,
(g) $P_{n+1}' - xP_n' - (n+1)P_n = 0$
(h) Expand $\sin \alpha x$ where $|\alpha| \leq \pi$ and $x \in [-1,1]$ in terms of the Legendre polynomials. Find the first five terms and discuss the convergence of the series expansion

$$\sum t^n P_n(x) = \frac{1}{\sqrt{1+t^2-2xt}}$$

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3. Find and prove the following for the Hermite Polynomials by using its generating function where $a_n = \frac{1}{n!}$:
(a) $H_n(-x) = (-1)^n H_n(x)$, (b) $||H_n||$, (c) $H_n(0)$
(d) $H_{n+1} - 2xH_n = 2nH_{n-1}$, (e) $H_n'' - 2xH_n' + 2nH_n = 0$,
(f) $\frac{d^m}{dx^m} H_n = 2^m \frac{n!}{(n-m)!} H_{n-m}$ (Use the Rodriguez formula to prove this property of the Hermite polynomial).
(g) Expand $e^{-\alpha x^2}$, $\alpha > 0$ in terms of the Hermite polynomials. Find the first five terms. Discuss the convergence of series expansion.

4. Prove the following for the Laguerre Polynomials:
(a) $nL_n^\nu - (n+\nu)L_{n-1}^\nu - xL_n^{\nu\prime} = 0$,
(b) $(n+1)\,L_{n+1}^\nu - (2n+\nu+1-x) + (n+\nu)L_{n-1}^\nu = 0$,
(c) Use the generating function to show that $L_n^\nu(0) = \Gamma(n+\nu+1)/[n!\Gamma(\nu+1)]$.
(d) Let $L_n(x) = L_m^0(x)$. Use the generating function for L_n and prove that $L_n'(0) = -n$, $L_n''(0) = \frac{1}{2}n(n-1)$
(e) Expand e^{-kx}, $k > 0$ as a series of Laguerre polynomials $L_n^\nu(x)$ Find the coefficients by using the orthogonality of L_n^ν and the generating function. Discuss the convergence of the series.

5. Prove that Orthogonal Polynomials $C_n(x)$ with $x \in [a, b]$ has n zeros in $[a, b]$.

6. Find $< xC_n, C_m >$ where $C_n(x)$ is any one of the classical orthogonal polynomial with $x \in [a, b]$

7. Express the function $f(x) = -1$ for $-1 \le x < 0$ and $f(x) = 1$ for $0 < x \le 1$ in terms of the Legendre polynomials. Find the first five terms and discuss the convergence of the series.

8. For the following problems remember the following theorem:

Theorem: *The Fourier series of a function $f(x)$ that is piecewise continuous in the interval $(-\pi, \pi)$ converges to*

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$$\frac{1}{2}[f(x+0)+f(x-0)] \quad for \quad -\pi < x < \pi, \tag{2}$$

$$\frac{1}{2}[f(\pi)+f(-\pi)] \quad for \quad x = \pm\pi \tag{3}$$

(a) Show that

$$|\sin \alpha x| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos 2kwx}{4k^2-1}$$

Discuss the above theorem for this example

(b) Find the Fourier series expansion of the function $f(x) = |x|$ in the interval $[-a.a]$. Discuss the above theorem for this example.

(c) Find the Fourier series expansion of the function $f(x) = x + a$ for $-a \leq x < 0$ and $f(x) = x$ for $0 < x \leq a$ and discuss the convergence of the series at the discontinuous points.

9. Prove that the function $f(x) = x$ for $0 \leq x \leq a$ and $f(x) = 2a - x$ for $a \leq x \leq 2a$ has Fourier representation

$$f(x) = \frac{8a^2}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin(n\pi/2)}{n^2}\sin\frac{n\pi x}{2a} = \frac{8a}{\pi^2}(\sin\frac{\pi x}{2a} - \frac{1}{3^2}\sin\frac{3\pi x}{2a} + \cdots)$$

Discuss the differentiability of the Fourier series. Compare the derivative function f' with the derivative of the Fourier series.

10. If the Fourier expansion

$$f(x) = A_0 + \sum_{n=1}^{\infty}(A_n\cos\frac{n\pi x}{a} + B_n\sin\frac{n\pi x}{a}) \quad (-a < x < a)$$

show that

$$\frac{1}{a}\int_{-a}^{a}[f(x)]^2dx = 2A_0^2 + \sum_{n=1}^{\infty}(A_n^2 + b_n^2)$$

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11. Prove the following

$$\cos\alpha x = \frac{\sin\pi\alpha}{\pi\alpha} + \sum_{n=1}^{\infty}(-1)^n \frac{2\alpha\sin\pi\alpha}{\pi(\alpha^2-n^2)}\cos nx$$

for $-\pi \leq x \leq \pi$. Here α is a non-integer real number. Deduce from this the following formula

$$\cot\pi\alpha = \frac{1}{\pi}[\frac{1}{\alpha} - \sum_{n=1}^{\infty}\frac{2\alpha}{n^2-\alpha^2}]$$

12. Prove that for $-\pi < x < \pi$

$$e^x = \frac{\sinh\pi}{\pi}[1 - 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2+1}(\cos nx - n\sin nx)]$$

From this result find the Fourier expansion of $\sinh x$ and $\cosh x$ in the same interval

13. (a). Using the Fourier Transform try to give a way to solve the following type of ODE: $y''(x) + ay'(x) + cy(x) = f(x)$ where $y'(0) = \alpha$ and $y(0) = \beta$. Here a, b, c are constants and $f(x)$ is a given function of x.
(b) Find the Fourier Transforms of $1/(1+x^2)$, $1/(1+x^2)^2$.